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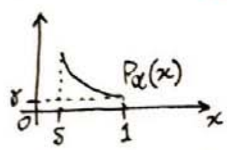
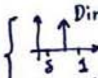
ESTIMATION OF BRADLEY-TERRY MODEL PARAMETERS: (with Ali Jadbabaie & Devavrat Shah)

① MOTIVATION:

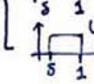
- Measure the level of skill in sports
  - [Getty et al 2018]: Luck vs Skill in fantasy sports ← motivated by policy-making on gambling
  - [Misra-Shah-Ranganathan 2020]: Hypothesis testing for pure skill
- Statistical Formulation:
  - Fix constants $\delta \in (0,1)$ and $\gamma, \sigma > 0$.
  - Let $\mathcal{P}_{\delta,\gamma}(\sigma)$ = set of all probability density functions (PDFs) on $[\delta, 1]$ that are $\geq \gamma$ and σ-Lipschitz continuous, i.e. $P_\alpha \in \mathcal{P}_{\delta,\gamma}(\sigma) \Leftrightarrow P_\alpha$ is a PDF and $\forall x, y \in [\delta, 1]$, $|P_\alpha(x) - P_\alpha(y)| \leq \sigma |x - y|$, $\forall x \in [\delta, 1]$, $P_\alpha(x) \geq \gamma$.
    non-parametric class
  - Each sport has an unknown PDF $P_\alpha \in \mathcal{P}_{\delta,\gamma}(\sigma)$ of merit values.



  - Suppose there is a tournament with $n \geq 2$ players: $\{1, \dots, n\}$.
    Each player i has merit value $\alpha_i \sim P_\alpha$ so that $\alpha_1, \dots, \alpha_n \overset{iid}{\sim} P_\alpha$.
  - There are $\binom{n}{2}$ independent two-player games in a tournament, with likelihoods:
$$\underbrace{\mathbb{P}(j \text{ beats } i \mid \alpha_1, \dots, \alpha_n) = \frac{\alpha_j}{\alpha_i + \alpha_j}}_{\text{This is the Bradley-Terry Model!}} \leftarrow \text{scale invariance} \quad \text{for all } i \neq j.$$
  - We see observations: $\{Z(i,j) \triangleq \mathbb{1}\{j \text{ beats } i\} : 1 \leq i < j \leq n\}$. (For $i > j$, $Z(i,j) = 1 - Z(j,i)$, and $Z(i,i) = 0$.)
  - GOAL: Estimate P_α or $h(P_\alpha) \triangleq -\int_\delta^1 P_\alpha(t) \log[P_\alpha(t)]\, dt$ from observations.
    ↑ differential entropy
    - Dirac measure $\Rightarrow h(P_\alpha) = -\infty$ ← Pure luck



    - Uniform measure $\Rightarrow h(P_\alpha) = \log(1-\delta)$ ← Pure skill



- Approach:
  Focus of Talk
  1) Estimate Bradley-Terry Model parameters $\alpha_1, \dots, \alpha_n$ based on observations.
     Suppose estimates are $\hat{\alpha}_1, \dots, \hat{\alpha}_n$.
  2) Estimate P_α or $h(P_\alpha)$ using $\hat{\alpha}_1, \dots, \hat{\alpha}_n$.
     (Robust kernel density estimation ← use $\hat{\alpha}_1, \dots, \hat{\alpha}_n$ instead of $\alpha_1, \dots, \alpha_n$.)
     ↳ $\hat{P}_\alpha(x) = \frac{1}{n} \sum_{i=1}^n \frac{1}{h} K\left(\frac{x - \hat{\alpha}_i}{h}\right)$ } Parzen-Rosenblatt estimator
     (h: bandwidth, K: kernel)

② BRADLEY-TERRY MODEL: MINIMAX ESTIMATION

- [Bradley-Terry 1952] (originally proposed by [Zermelo 1929]): Ranking based on paired comparisons.
  - n items $\{1, \dots, n\}$ with underlying merits $\alpha_1, \dots, \alpha_n \geq 0$
  - Easy to compare any two, but hard to rank all.
  - Use model $\mathbb{P}(i > j) = \frac{\alpha_i}{\alpha_i + \alpha_j}$ of pairwise comparisons to find "true" merits $\alpha_1, \dots, \alpha_n$.

[continued.]

- Plackett-Luce Model: [Luce 1959], [Plackett 1975] ← social choice theory/econometrics

  - Luce's choice axiom: Probability of selecting one item over another in a set of items is not affected by the presence or absence of other items in the set. ↑ axiom for prob. model over selection

    ↑ independence of irrelevant alternatives

  - Equivalent model: n items $\{1,\dots,n\}$ with merits $\alpha_1,\dots,\alpha_n \geq 0$

    $$\mathbb{P}(\text{select } i) = \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}$$

  - Distribution over permutations/rankings:

    $$\forall \sigma, \quad \underset{\uparrow \text{ permutation of } \{1,\dots,n\}}{\mathbb{P}_{PL}(\sigma)} = \left.\frac{\alpha_{\sigma(1)}}{\sum_{k\in\{1,\dots,n\}} \alpha_k} \cdot \frac{\alpha_{\sigma(2)}}{\sum_{k\in\{1,\dots,n\}\setminus\{\sigma(1)\}} \alpha_k} \cdot \frac{\alpha_{\sigma(3)}}{\sum_{k\in\{1,\dots,n\}\setminus\{\sigma(1),\sigma(2)\}} \alpha_k} \cdots\cdots \frac{\alpha_{\sigma(n)}}{\alpha_{\sigma(n)}}\right\} \text{Plackett-Luce model}$$

  - Pairwise selection → Bradley-Terry model

- Thurstonian Model: [Thurstone 1927] ← psychometrics

  - Law of Comparative Judgment: "Discriminal" process to rank n items $\{1,\dots,n\}$ is modeled by first associating merits $\alpha_1,\dots,\alpha_n \geq 0$ to the items, and then ranking them by ranking the n random variables $\alpha_1 + X_1, \dots, \alpha_n + X_n$ for i.i.d. $\underbrace{X_1,\dots,X_n}_{\text{noise in discriminal process}}$.

  - Distribution over permutations/rankings:

    $$\forall \sigma, \quad \underset{\uparrow \text{ permutation of } \{1,\dots,n\}}{\mathbb{P}_T(\sigma)} = \mathbb{P}\left(\alpha_{\sigma(1)} + X_{\sigma(1)} > \alpha_{\sigma(2)} + X_{\sigma(2)} > \cdots > \alpha_{\sigma(n)} + X_{\sigma(n)}\right)$$

  - Equivalent to Plackett-Luce model if and only if $X_1,\dots,X_n \overset{iid}{\sim} \text{Gumbel}(\mu, \beta)$. [Yellott 1977]

    generalized extreme value (Type I) dist. — μ: ↑ real location; β: positive scale

    (Note: $F_X(x) \triangleq e^{-e^{-(x-\mu)/\beta}}$ is CDF of Gumbel dist.)

- Minimax Formulation:

  Define $\pi(i) \triangleq \dfrac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}$, $\forall i \in \{1,\dots,n\}$, and let $\pi \triangleq (\pi(1),\dots,\pi(n))$.

  ↑ canonically scaled merit parameters

  Find upper and lower bounds on:

  $$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}_{\delta,\beta}(\sigma)} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right]$$

  ← ℓ^1-norm

  ↑ wrt $P_\alpha^{\otimes n}, P_{Z|\alpha_1,\dots,\alpha_n}$

  where infimum is over all estimators $\hat{\pi}$ of π based on $\{Z(i,j) : i < j\}$.

  ↑ could be randomized, must be a prob. mass function

[continued.]

③ MINIMAX UPPER BOUND: ← construct estimator

- Rank Centrality: [Negahban-Oh-Shah 2017]

- Define the row stochastic matrix $S \in \mathbb{R}^{n\times n}$:

$$\forall i\neq j,\quad \underbrace{S(i,j)}_{(i,j)\text{th entry of } S} \triangleq \frac{1}{n-1}\cdot\frac{\alpha_j}{\alpha_i+\alpha_j} \overset{\text{since } \alpha_i\text{'s} \geq \delta}{>} 0,$$

$$\forall i,\quad S(i,i) \triangleq 1-\frac{1}{n-1}\sum_{k\neq i}\frac{\alpha_k}{\alpha_i+\alpha_k} = \frac{1}{n-1}\sum_{k\neq i}\frac{\alpha_i}{\alpha_i+\alpha_k} \overset{\text{since } \alpha_i\text{'s} \geq \delta}{>} 0.$$

(Clearly, $\sum_{j=1}^{n} S(i,j) = 1,\ \forall i.$)

- S defines a Markov chain on the state-space of players $\{1,\dots,n\}$.

- Detailed Balance Conditions:

$$\forall i\neq j,\quad \pi(i)S(i,j) = \frac{\alpha_i}{\sum_{k=1}^{n}\alpha_k}\cdot\frac{1}{n-1}\cdot\frac{\alpha_j}{\alpha_i+\alpha_j} = \frac{\alpha_j}{\sum_{k=1}^{n}\alpha_k}\cdot\frac{1}{n-1}\cdot\frac{\alpha_i}{\alpha_i+\alpha_j} = \pi(j)S(j,i).$$

S self-adjoint operator

S satisfies Kolmogorov criterion

Hence, S defines a reversible Markov chain with invariant distribution π: $\pi = \pi S$.

- π is unique as $S>0$ entry-wise $\Rightarrow$ S ergodic, i.e. irreducible & aperiodic.

- Construct estimator $\tilde{S}\in\mathbb{R}^{n\times n}$ of S based on $Z(i,j)$'s:

$$\left.\begin{aligned}
&\forall i\neq j,\quad \tilde{S}(i,j) \triangleq \frac{1}{n-1}Z(i,j) \geq 0,\\
&\forall i,\quad \tilde{S}(i,i) \triangleq 1-\frac{1}{n-1}\sum_{k\neq i}Z(i,k) = \frac{1}{n-1}\sum_{k\neq i}Z(k,i) \geq 0.\\
&\left(\text{Clearly, } \sum_{j=1}^{n}\tilde{S}(i,j) = 1,\ \forall i.\right)
\end{aligned}\right\}\ \tilde{S} \text{ is row stochastic}$$

- $\tilde{S}$ defines another Markov chain. ← not necessarily reversible or ergodic

Let $\tilde{\pi} = \tilde{\pi}\tilde{S}$ be any invariant distribution of $\tilde{S}$.

- $\boxed{\tilde{\pi} \text{ is an estimator of } \pi}$.

Thm: [Chen et al 2019]

a) $\dfrac{\|\tilde{\pi}-\pi\|_\infty}{\|\pi\|_\infty} = O\left(\frac{1}{\delta}\sqrt{\frac{\log(n)}{n}}\right)$ with probability $\geq 1-O(n^{-5})$.

b) $\dfrac{\|\tilde{\pi}-\pi\|_2}{\|\pi\|_2} = \underbrace{O\left(\frac{1}{\sqrt{n}}\right)}_{\text{if } \delta=\Theta(1)}$ with probability $\geq 1-O(n^{-5})$.

- Can we bound $\|\tilde{\pi}-\pi\|_1$ with high probability?

Yes!

[continued.]

Anuran Makur

- **Theorem:** (Upper Bound) $$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}_{\delta,\gamma}(\sigma)} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right] \leq \sup_{P_\alpha \in \mathcal{P}_{\delta,\gamma}(\sigma)} \mathbb{E}\left[\|\tilde{\pi} - \pi\|_1\right] = O\left(\frac{1}{\sqrt{n}}\right)$$ for all sufficiently large n.

- Proof:

Using [Chen et al 2019] part (b), with probability $\geq 1 - O(n^{-5})$,

$$\|\tilde{\pi} - \pi\|_1 \leq \sqrt{n}\, \|\tilde{\pi} - \pi\|_2 \leq \underbrace{C}_{\text{some constant}} \|\pi\|_2$$

for all sufficiently large n, using equivalence of norms.

Since $\|\pi\|_2 \leq \frac{1}{\underbrace{\delta}_{\text{constant}}\sqrt{n}}$ (because $\alpha_i \in [\delta, 1]$), we get:

$$\|\tilde{\pi} - \pi\|_1 = O\left(\frac{1}{\sqrt{n}}\right) \text{ with probability } \geq 1 - O(n^{-5}).$$

The law of total expectation and the bound $\|\tilde{\pi} - \pi\|_1 \underset{\text{triangle inequality}}{\leq} \|\tilde{\pi}\|_1 + \|\pi\|_1 = 2$ yield the desired result. ▨

④ Minimax Lower Bound:

- **Bayes Risk:**

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}_{\delta,\gamma}(\sigma)} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right] \geq \overbrace{\inf_{\hat{\pi}} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right]}^{\text{Bayes risk}}$$

Choose $P_\alpha = \text{Uniform}([\delta, 1])$.

How do we lower bound Bayes risk?

- **Generalized Fano's Method:**

- Lemma: [Xu-Raginsky 2017] $$\inf_{\hat{\pi}} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right] \geq \sup_{t > 0} t\left(1 - \frac{I(\alpha_1^n; Z) + \log(2)}{\log(1/\mathcal{L}(t))}\right)$$

$\alpha_1^n \triangleq (\alpha_1, \ldots, \alpha_n)$, $Z \triangleq \{Z(i,j) : i < j\}$

where $\underbrace{I(\alpha_1^n; Z)}_{\text{mutual information}} \triangleq \underbrace{D}_{\text{KL divergence}}(P_{\alpha_1^n, Z} \| P_{\alpha_1^n} P_Z)$ and $\mathcal{L}(t) \triangleq \underbrace{\sup_{\nu} \mathbb{P}\left(\|\pi - \nu\|_1 \leq t\right)}_{\text{small ball probability (measure of concentration)}}$ for $t > 0$.

- Intuition: $\mathcal{L}(t) \uparrow \Rightarrow$ high conc. of $\pi \Rightarrow$ Bayes risk $\downarrow$ (as we can estimate π easily).
$I(\alpha_1^n; Z) \uparrow \Rightarrow Z$ has lots of info. about $\pi \Rightarrow$ Bayes risk $\downarrow$.

[continued.]

- Proof: Fix any $\hat{\pi}$ and any $t>0$.
  Consider Markov chain $\pi \to \alpha_1^n \to Z \to \hat{\pi}$.

$$\begin{aligned}
I(\alpha_1^n;Z) &\geq I(\pi;\hat{\pi}) \quad [\text{DPI}] \\
&= D(P_{\pi,\hat{\pi}}\|P_\pi \cdot P_{\hat{\pi}}) = D(P_{\pi,\hat{\pi}}\|Q_{\pi,\hat{\pi}}), \text{ where } Q_{\pi,\hat{\pi}} \triangleq P_\pi \cdot P_{\hat{\pi}} \\
&\geq \underbrace{D(P_{\pi,\hat{\pi}}(\|\pi-\hat{\pi}\|_1 \leq t)\|Q_{\pi,\hat{\pi}}(\|\pi-\hat{\pi}\|_1\leq t))}_{\text{binary KL divergence}} \quad [\text{DPI: } (\pi,\hat{\pi}) \mapsto \mathbb{1}\{\|\hat{\pi}-\pi\|_1 \leq t\}] \\
&= D\big(P_{\pi,\hat{\pi}}(\|\pi-\hat{\pi}\|_1 \leq t)\,\big\|\,\mathbb{E}_{\hat{\pi}}[P_\pi(\|\pi-\hat{\pi}\|_1\leq t)]\big) \\
&\geq P_{\pi,\hat{\pi}}(\|\pi-\hat{\pi}\|_1 \leq t)\log\left(\frac{1}{\mathbb{E}_{\hat{\pi}}[P_\pi(\|\pi-\hat{\pi}\|_1\leq t)]}\right) - \log(2) \\
&\geq \mathbb{P}(\|\pi-\hat{\pi}\|_1 \leq t)\log\left(\frac{1}{\mathcal{L}(t)}\right) - \log(2).
\end{aligned}$$

Lemma: $\forall p,q \in [0,1]$,
$$\begin{aligned}
D(p\|q) &= p\log\left(\frac{p}{q}\right) + (1-p)\log\left(\frac{1-p}{1-q}\right) \\
&= p\log\left(\frac{1}{q}\right) + \underbrace{(1-p)\log\left(\frac{1}{1-q}\right)}_{\geq 0} - h(p) \quad \leftarrow \text{binary entropy} \\
&\geq p\log\left(\frac{1}{q}\right) - \log(2).
\end{aligned}$$

$$\Rightarrow \mathbb{P}(\|\hat{\pi}-\pi\|_1 > t) \geq 1 - \frac{I(\alpha_1^n;Z)+\log(2)}{\log\left(\frac{1}{\mathcal{L}(t)}\right)}.$$

By Markov's inequality:
$$\begin{aligned}
\mathbb{E}[\|\hat{\pi}-\pi\|_1] &\geq t\,\mathbb{P}(\|\hat{\pi}-\pi\|_1 > t) \\
&\geq t\left(1 - \frac{I(\alpha_1^n;Z)+\log(2)}{\log(1/\mathcal{L}(t))}\right).
\end{aligned}$$
Take $\inf_{\hat{\pi}}$ and $\sup_{t>0}$. This completes the proof. ∎

- Bound $I(\alpha_1^n;Z)$: (Covering Number Method)

  - For $\beta = (\beta_1,\dots,\beta_n) \in [\delta,1]^n$, let $P_{Z|\beta} = P_{Z|\alpha_1^n=\beta}$.
  - Def: For any $\varepsilon>0$, we say that $\{\beta^{(1)},\dots,\beta^{(M)}\} \subset [\delta,1]^n$ is an ε-covering of $[\delta,1]^n$ with cardinality M if $\forall \beta \in [\delta,1]^n$, $\exists i \in \{1,\dots,M\}$, $D(P_{Z|\beta}\|P_{Z|\beta^{(i)}}) \leq \varepsilon$.

    $M^*(\varepsilon) \triangleq \min\{M : \exists\, \varepsilon\text{-covering with cardinality } M\}$.

    ↑ ε-covering number
  - Lemma: [Yang-Barron 1999] $I(\alpha_1^n;Z) \leq \inf_{\varepsilon>0} \varepsilon + \log(M^*(\varepsilon))$.

  Proof: Fix any $\varepsilon>0$. Let $\{\beta^{(1)},\dots,\beta^{(\underbrace{M^*(\varepsilon)}_{=M^*})}\}$ be an ε-covering. Let $i(\beta) \triangleq \operatorname*{argmin}_{i\in\{1,\dots,\underbrace{M^*(\varepsilon)}_{=M^*}\}} D(P_{Z|\beta}\|P_{Z|\beta^{(i)}})$, $\forall \beta \in [\delta,1]^n$.

$$\begin{aligned}
I(\alpha_1^n;Z) &= D(P_{\alpha_1^n,Z}\|P_{\alpha_1^n}\cdot P_Z) \\
&= \mathbb{E}_{P_{\alpha_1^n,Z}}\left[\log\left(\frac{P_{\alpha_1^n,Z}}{P_{\alpha_1^n}\cdot P_Z}\right)\right] = \mathbb{E}\left[\log\left(\frac{P_{Z|\alpha_1^n}\cdot \frac{1}{M^*}\sum_{i=1}^{M^*} P_{Z|\beta^{(i)}}}{P_Z \cdot \frac{1}{M^*}\sum_{i=1}^{M^*} P_{Z|\beta^{(i)}}}\right)\right] \\
&= \mathbb{E}_{\alpha_1^n}\left[D\left(P_{Z|\alpha_1^n}\,\Big\|\,\frac{1}{M^*}\sum_{i=1}^{M^*}P_{Z|\beta^{(i)}}\right)\right] - \underbrace{D\left(P_Z\,\Big\|\,\frac{1}{M^*}\sum_{i=1}^{M^*}P_{Z|\beta^{(i)}}\right)}_{\geq 0 \text{ [Gibbs]}} \\
&\leq \mathbb{E}_{\alpha_1^n}\left[\mathbb{E}_{P_{Z|\alpha_1^n}}\left[\log\left(\frac{P_{Z|\alpha_1^n}}{\frac{1}{M^*}P_{Z|\beta^{(i(\alpha_1^n))}}}\right)\right]\right] \\
&= \mathbb{E}_{\alpha_1^n}\Big[\underbrace{D(P_{Z|\alpha_1^n}\|P_{Z|\beta^{(i(\alpha_1^n))}})}_{\leq \varepsilon \text{ [}\varepsilon\text{-covering]}}\Big] + \log(M^*(\varepsilon)) \\
&\leq \varepsilon + \log(M^*(\varepsilon)).
\end{aligned}$$

Take $\inf_{\varepsilon>0}$. (Note: Can also take $\sup_{P_{\alpha_1^n}}$.) ∎

↑ bound on channel capacity

[continued.]

- **Lemma:** $I(\alpha_1^n; Z) \leq \overbrace{\frac{1}{2}n\log(n) + \frac{(1-\delta)^2}{8\delta^2}\left(2+\delta+\frac{1}{\delta}\right)n}^{O(n\log(n))}$.

Proof: Let $Q \triangleq \left\{\delta + \frac{(1-\delta)k}{\sqrt{n}} : k \in \{1,\ldots,\lfloor\sqrt{n}\rfloor\}\right\}$. ← quantize $[\delta,1]$

floor function

Then, $\forall t \in [\delta,1]$, $\min_{s\in Q} |t-s| \leq \frac{1-\delta}{\sqrt{n}}$.

Claim: Q^n is an ε-covering with $|Q^n| = |Q|^n \leq n^{n/2}$ and $\varepsilon = \frac{(1-\delta)^2}{8\delta^2}\left(2+\delta+\frac{1}{\delta}\right)n$.

→ Pf: Fix any $\beta = (\beta_1,\ldots,\beta_n) \in [\delta,1]^n$, and choose $\gamma = (\gamma_1,\ldots,\gamma_n) \in Q^n$ such that $|\beta_i - \gamma_i| \leq \frac{1-\delta}{\sqrt{n}}$, $\forall i$.

$$\begin{aligned}
D(P_{Z|\beta}\,\|\,P_{Z|\gamma}) &= \sum_{i<j} D\left(P_{Z(i,j)|\beta}\,\|\,P_{Z(i,j)|\gamma}\right) \\
&= \sum_{i<j} D\left(\frac{\beta_j}{\beta_i+\beta_j}\,\Big\|\,\frac{\gamma_j}{\gamma_i+\gamma_j}\right) \quad \leftarrow \text{binary KL divergence} \\
&\leq \sum_{i<j} \chi^2\left(\frac{\beta_j}{\beta_i+\beta_j}\,\Big\|\,\frac{\gamma_j}{\gamma_i+\gamma_j}\right) \quad \leftarrow \text{binary } \chi^2\text{-divergence} \\
&= \sum_{i<j}\left(\frac{\beta_j}{\beta_i+\beta_j} - \frac{\gamma_j}{\gamma_i+\gamma_j}\right)^2\left(2 + \frac{\gamma_i}{\gamma_j} + \frac{\gamma_j}{\gamma_i}\right) \\
&\leq \underbrace{\left(2+\delta+\frac{1}{\delta}\right)}_{\substack{[\delta,\frac{1}{\delta}]\ni t \mapsto t+\frac{1}{t} \\ \text{maximized at endpoints}}} \sum_{i<j} \underbrace{\left(\left|\frac{\beta_j}{\beta_i+\beta_j} - \frac{\beta_j}{\gamma_i+\beta_j}\right| + \left|\frac{\beta_j}{\gamma_i+\beta_j} - \frac{\gamma_j}{\gamma_i+\gamma_j}\right|\right)^2}_{\Delta\text{-inequality}} \\
&\leq \left(\frac{1}{4\delta}\right)^2\left(2+\delta+\frac{1}{\delta}\right)\sum_{i<j}\underbrace{\left(|\beta_i-\gamma_i| + |\beta_j-\gamma_j|\right)^2}_{\substack{F:[\delta,\infty)^2\to\mathbb{R},\ F(x,y)\triangleq\frac{x}{x+y} \\ \cdot\text{ For fixed } x,\ F \text{ is } \frac{1}{4\delta}\text{-Lipschitz in } y. \\ \cdot\text{ For fixed } y,\ F \text{ is } \frac{1}{4\delta}\text{-Lipschitz in } x.}} \\
&\leq \frac{(1-\delta)^2}{8\delta^2}\left(2+\delta+\frac{1}{\delta}\right)(n-1) \\
&\leq \varepsilon.
\end{aligned}$$

//

Using Yang-Barron Lemma,

$$\begin{aligned}
I(\alpha_1^n; Z) &\leq \overset{\text{our } \varepsilon \text{ value}}{\varepsilon} + \log(M^*(\varepsilon)) \\
&\leq \varepsilon + \log(|Q^n|) \quad \leftarrow \text{our } \varepsilon\text{-covering} \\
&\leq \frac{1}{2}n\log(n) + \frac{(1-\delta)^2}{8\delta^2}\left(2+\delta+\frac{1}{\delta}\right)n.
\end{aligned}$$

This completes the proof. ▨

- **Remark:** This is better than standard information inequalites (tensorization bounds), which give $I(\alpha_1^n; Z) = O(n^2)$.

[continued.]

- **Bound Small Ball Probability:** ← no standard approach in the literature

  - **Lemma:** $\forall t > 0, \quad \mathcal{L}(t) \leq \left(\frac{2e}{1-\delta}\right)^n t^{n-1}$.

    Proof: For any $t > 0$,

$$
\begin{aligned}
\mathcal{L}(t) &= \sup_{\nu} \mathbb{P}\left(\|\pi - \nu\|_1 \leq t\right) \\
&\leq \sup_{\tilde{\nu}} \mathbb{P}\left(\|\tilde{\pi} - \tilde{\nu}\|_1 \leq t\right) \quad \leftarrow \begin{array}{l} \tilde{\pi} = (\pi(1), \dots, \pi(n-1)) \\ \tilde{\nu} = (\nu_1, \dots, \nu_{n-1}) \text{ where } \nu = (\nu_1, \dots, \nu_n) \end{array} \\
&= \sup_{\tilde{\nu}} \int_{\mathbb{B}^{n-1}} \underbrace{P_{\tilde{\pi}}(\tau)}_{\text{PDF of } \tilde{\pi} \text{ (Note: } \pi \text{ has degenerate PDF.)}} \mathbb{1}\{\|\tau - \tilde{\nu}\|_1 \leq t\}\, d\tau \\
&\leq \left(\sup_{\tau \in \mathbb{B}^{n-1}} P_{\tilde{\pi}}(\tau)\right) \cdot \underbrace{\mathrm{vol}\left(\{x \in \mathbb{B}^{n-1} : \|x\|_1 \leq t\}\right)}_{\text{volume of } \ell^1\text{-ball in } \mathbb{B}^{n-1} \text{ with radius } t \,=\, \frac{2^{n-1}}{(n-1)!} t^{n-1}} \\
&= \frac{2^{n-1}}{(n-1)!} t^{n-1} \cdot \sup_{\tau \in \mathbb{B}^{n-1}} P_{\tilde{\pi}}(\tau) \\
&\leq \frac{2^{n-1}}{(n-1)!} t^{n-1} \cdot \frac{n^{n-1}}{(1-\delta)^n} \\
&= \frac{1}{2} \cdot \frac{n^n}{n!} \left(\frac{2}{1-\delta}\right)^n t^{n-1} \\
&\leq \frac{1}{5\sqrt{n}} \left(\frac{2e}{1-\delta}\right)^n t^{n-1} \\
&\leq \left(\frac{2e}{1-\delta}\right)^n t^{n-1} .
\end{aligned}
$$

Lemma: $\sup_{\tau \in \mathbb{B}^{n-1}} P_{\tilde{\pi}}(\tau) \leq \frac{n^{n-1}}{(1-\delta)^n}$

(Follows from explicit calculation of PDF $P_{\tilde{\pi}}$ from $P_{\alpha_i^n}$ based on change-of-var. formula.)

Stirling's formula: $n! \geq \frac{5}{2}\sqrt{n}\, \frac{n^n}{e^n}$.

This completes the proof. ▨

- **Theorem:** (Lower Bound) For any $\varepsilon > 0$,

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}_{\delta,\sigma}} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right] \geq \left(\frac{\varepsilon}{4 + 2\varepsilon}\right) \frac{1}{n^{\frac{1}{2} + \varepsilon}}$$

for all $n \geq 2$ sufficiently large (depending on ε, δ).

↑ $n\uparrow$ as $\varepsilon \downarrow 0$, $n \uparrow$ as $\delta \downarrow 0$

  - **Remark:** For any $\varepsilon > 0$, and all $n \in \mathbb{N}$ sufficiently large:

$$\Omega\left(\frac{1}{n^{\frac{1}{2}+\varepsilon}}\right) \leq \inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}_{\delta,\sigma}} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right] \leq O\left(\frac{1}{\sqrt{n}}\right) .$$

[continued.]

- Proof: Fix any $\varepsilon > 0$. Using previous lemmata:

$$\inf_{\hat{\pi}} \sup_{P_\alpha \in \mathcal{P}_{\delta,\gamma}(\sigma)} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right] \geq \inf_{\hat{\pi}} \mathbb{E}\left[\|\hat{\pi} - \pi\|_1\right]$$

$P_\alpha = \text{Uniform}([\delta, 1])$

$$\geq \sup_{t>0} t\left(1 - \frac{I(\alpha_1^n; Z) + \log(2)}{\log(1/\mathcal{L}(t))}\right) \quad \text{[Generalized Fano Lemma]}$$

$$\geq \sup_{t>0} t\left(1 - \frac{\frac{1}{2}n\log(n) + \frac{(1-\delta)^2}{8\delta^2}\left(2+\delta+\frac{1}{\delta}\right)n + \log(2)}{(n-1)\log(1/t) - \log(2e/(1-\delta))\,n}\right) \quad \text{[Upper Bounds on } I(\alpha_1^n; Z) \text{ and } \mathcal{L}(t)\text{]}$$

$$\geq \sup_{t>0} t\left(1 - \frac{1 + O\left(\frac{1}{\log(n)}\right)}{\frac{2(n-1)\log(1/t)}{n\log(n)} - O\left(\frac{1}{\log(n)}\right)}\right)$$

$\boxed{\text{Set } t = \frac{1}{n^{\frac{1}{2}+\varepsilon}}}$

$$\geq \frac{1}{n^{\frac{1}{2}+\varepsilon}}\underbrace{\left(1 - \frac{1 + O(1/\log(n))}{(1+2\varepsilon)\left(1-\frac{1}{n}\right) - O(1/\log(n))}\right)}_{\to\, 1 - \frac{1}{1+2\varepsilon} \,=\, \frac{2\varepsilon}{1+2\varepsilon} \text{ as } n\to\infty}$$

$$\geq \frac{1}{n^{\frac{1}{2}+\varepsilon}}\left(\frac{\varepsilon}{4+2\varepsilon}\right) \quad \text{for } n \geq 2 \text{ sufficiently large.}$$

This completes the proof. ∎